Model Conversion and Digital Redesign of Singular Systems

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ABSTRACT: *Design procedures are proposed for model conversion and digital redesign of a singular system, which is controllable at finite and impulsive modes. In order to attain a standard regular problem, we use some techniques to decompose the singular system into a reduced-order regular subsystem and a nondynamic subsystem. As a result, some well-known design methodologies for a regular system can be applied to the reduced-order regular subsystem. Finally, we transform the results obtained back to those of the original coordinate system.*

I. Introduction

A large number of control systems are characterized by continuous-time dynamic equations. Also many theories and practical methods have been developed for continuous-time models. However, it is well-known that since the digital computers and digital processors are not only greatly advanced in technology, but also possess many advantages such as improved sensitivity, better reliability, no drift, less effect due to noise as well as disturbance, lower cost, etc., so it is often desirable to refit these systems with digital transducers and digital controllers. In order to match the states of the equivalent discrete-time system to those of the continuous-time system as closely as possible, the sampling period must be sufficiently small. Unfortunately, the resulting discrete-time system may be unstable even if the original continuous-time system is stable. Thus, we apply the so-called digital redesign technique (**1**–**3**) to arrive at an approximated digital system which closely matches the response of the continuous-time system with the same inputs and initial conditions, rather than designing a new sytsem using digital control theory.

In this paper, we extend the model conversion and digital redesign problems to singular systems, also called descriptor systems, characterized by

$$E_r\dot{x}(t) = A_r x(t) + B_r u(t), \tag{1}$$

where $x(t) \in R^n$, $u(t) \in R^m$, and E_r is a singular matrix. The constant matrices E_r, A_r, as well as B_r, all have appropriate dimensions. It is known that singular systems are of practical importance since they appear in many areas such as electrical networks, singularly perturbed systems, composite systems, Leontieff models in multisector economy, Leslie population models in biology, etc. (**4**).

It should be noted that we assume the pencil $(sE_r - A_r)$ to be regular, i.e. $\det(sE_r - A_r) \neq 0$ and that the singular system has a unique solution if the singular system considered in (1) is controllable at finite and impulsive modes in the sense of Cobb (**5**).

Definition 1 (**5**)

The singular system is referred to as controllable at finite modes iff rank $[sE_r - A_r, B_r] = n$, for all finite $s \in C$.

Definition 2 (**5**)

If rank $[E_r, B_r] = n$, then the singular system is referred to as controllable at impulsive modes.

It is also noted that even if rank $[E_r, B_r] \neq n$, there may still exist a dynamic feedback control law such that impulsive modes can be moved to finite locations. In addition, if rank $(E_r) - \deg\{\det(sE_r - A_r)\} \triangleq q$, where rank $(E_r) \triangleq g$ is called the generalized order of the singular system, and $\deg\{\det(sE_r - A_r)\} \triangleq k$ is said to be the order of the slow subsystem, then the singular system has q impulsive modes, which occur in the fast subsystem and are created by either inconsistent initial conditions or discontinuous control input (**6**–**7**).

Cobb (**8**) and Tsai *et al.* (**9**) used preliminary linear feedback controllers to make the singular system (1) causal (i.e. to remove impulsive modes from the system response by moving the associated poles from infinity to some finite locations). Tsai's work is more comparable to the elegant and technical paper proposed by Cobb (**8**), because Cobb's approach needs to solve complex and difficult eigenvalue–eigenvector problems, in particular, when the singular system has a large order and is in the deflective Jordan form. Also, it is not easy to determine a preliminary feeback control law such that the impulsive modes can be moved to finite locations. Although sometimes Tsai's method requires solution of the eigenvector problem, it is not complicated for processing a low-order submatrix with all zero eigenvalues.

In this paper, we first transform the singular system with a regular pencil (**10**) into a standard one (**11**) by using a block modal matrix, constructed via the fast and numerically stable algorithms of the extended matrix sign function (**12**–**13**). Thus, the singular system can be decomposed into a slow subsystem with finite modes, and a fast subsystem with infinite modes which include impulsive modes and nondynamic infinite modes. If the singular system has no impulsive modes, then the slow subsystem is in a form of a reduced-order regular system and the fast subsystem is in a nondynamic system in which the fast states only depend on the input. On the contrary, if the singular system has any impulsive modes, then we use the preliminary feedback control law to eliminate it, and repeat the above procedure so that the singular system can be decomposed into a reduced-order regular system and a nondynamic equation. As a result, we can apply the given results (**2**, **14**) with respect to model conversion and digital redesign to the reduced-order regular system. Finally, the results obtained can be transformed back to those of the original coordinate system by using similarity transformations and simple substitutions.

II. Preliminaries

(i) *Introduction to the matrix sign function*

The matrix sign function of a square matrix $A \in C^{n \times n}$ with $\mathrm{Re}(\sigma(A)) \neq 0$ is defined by (**15**)

$$\mathrm{sign}(A) = 2\,\mathrm{sign}^{+}(A) - I_n, \tag{2}$$

where I_n is an $n \times n$ identity matrix and

$$\mathrm{sign}^{+}(A) = \frac{1}{2\pi i} \oint_{c_+} (\lambda I_n - A)^{-1}\,\mathrm{d}\lambda. \tag{3}$$

c_+ is a simple closed contour in right-half plane of λ and encloses all the right-half-plane eigenvalues of A. On the other hand, the matrix sign function (**12**, **16**) is also defined as

$$\mathrm{sign}(A) = A(\sqrt{A^2})^{-1} = A^{-1}(\sqrt{A^2}), \tag{4}$$

where the matrix $\sqrt{A^2}$ denotes the principal square root of A^2. Two fast and stable algorithms (**12**, **13**) with $r(\leqslant 3)$th-order convergence rates to compute the matrix sign function are listed below.

For $r = 2$, one has

$$Q(k+1) = Q(k)[2I_n][I_n + Q^2(k)]^{-1}, \quad Q(0) = A, \quad \lim_{k \to \infty} Q(k) = \mathrm{sign}(A) \tag{5}$$

or

$$Q(k+1) = \tfrac{1}{2}[Q^{-1}(k) + Q(k)], \quad Q(0) = A, \quad \lim_{k \to \infty} Q(k) = \mathrm{sign}(A). \tag{6}$$

Note that $Q(k) = Q^{-1}(k)$.

For $r = 3$, one has

$$Q(k+1) = Q(k)[3I_n + Q^2(k)][I_n + 3Q^2(k)]^{-1}, \quad Q(0) = A, \quad \lim_{k \to \infty} Q(k) = \mathrm{sign}(A). \tag{7}$$

Some fast and numerically stable algorithms with $r \geqslant 4$ can be found in Shieh *et al.* (**12**) and Tsai *et al.* (**13**).

One main feature of the matrix sign function is that it preserves the eigenvectors of the original matrix. This property is useful for studying the eigenstructures of matrices, as well as for developing applications for engineering problems. A singular matrix A can be modified using the bilinear transformation,

$$\tilde{A} = (A - \rho I_n)(A + \rho I_n)^{-1}, \tag{8}$$

where ρ is the radius of a circle with center at the origin such that the circle contains only those zero eigenvalues, and no eigenvalue of A is located on the circle. Therefore, the eigenvalues within the circle will be mapped into the left-half plane of the complex s-plane, and those outside the circle will be mapped into the right-half plane of the complex s-plane. Thus, the proposed algorithms in (5)–(7) can be applied for obtaining the sign ($\tilde{A}$). Note that the bilinear transformation preserves the eigenvectors of the original system.

(ii) *Introduction to the regular pencil and the standard one*

Definition 3 (**10**)

Let E_r and A_r be two square constant matrices. If $\det(sE_r - A_r) \neq 0$, for all s, then $(sE_r - A_r)$ is called a regular pencil.

Definition 4 (**11**)

Let $(sE_n - A_n)$ be a regular pencil. If there exist scalars α and β such that $\alpha E_n + \beta A_n = I_n$, then $(sE_n - A_n)$ is called a standard pencil.

Note that for any regular linear system (with $E_r = I_n$), the regular pencil becomes a standard pencil by taking $\alpha = 1$ and $\beta = 0$. Furthermore, any regular pencil, $(sE_r - A_r)$, can be easily transformed into a standard one by premultiplying $(\alpha E_r + \beta A_r)^{-1}$ to E_r and A_r, respectively, where α and β are scalars such that $\det(\alpha E_r + \beta A_r) \neq 0$. Hence, the matrix coefficients of a standard pencil, $(sE_n - A_n)$, become

$$E_n \triangleq (\alpha E_r + \beta A_r)^{-1} E_r \tag{9}$$

$$A_n \triangleq (\alpha E_r + \beta A_r)^{-1} A_r. \tag{10}$$

The modified system retains its state vector $x(t)$ and the matrices E_n and A_n have the following properties.

Lemma 1 (**4**)

(L1) $E_n A_n = A_n E_n$, that is, E_n and A_n commute.
(L2) E_n and A_n have the same eigenspaces.

The above properties enable us to decompose a singular system into a reduced-order regular subsystem and a nondynamic subsystem. The detailed derivation is shown in the next section.

III. The Optimal Regional-pole-placement Design Method for Singular Systems (9)

Consider a linear continuous-time singular system, which is controllable at finite and impulsive modes, characterized by (1). It is known that the singular system can be decomposed into a slow and a fast subsystem. As we have discussed in Section II, the regular pencil $(sE_r - A_r)$ can easily be transformed into a standard one, say $(sE_n - A_n)$. Note that since E is a singular matrix, which has at least one zero eigenvalue, β can not equal zero. Hence, we have

$$(\alpha E_r+\beta A_r)^{-1}E_r\dot{x}(t) = (\alpha E_r+\beta A_r)^{-1}A_r x(t)+(\alpha E_r+\beta A_r)^{-1}B_r u(t)$$
$$\Rightarrow E_n\dot{x}(t) = A_n x(t)+B_n u(t), \quad (11)$$

where $E_n = (\alpha E_r+\beta A_r)^{-1}E_r$, $A_n = (\alpha E_r+\beta A_r)^{-1}A_r$ and $B_n = (\alpha E_r+\beta A_r)^{-1}B_r$. Due to $\alpha E_n+\beta A_n = I_n$, so the pencil (sE_n-A_n) is a standard one which has the properties mentioned in Lemma 1.

Now, let

$$x(t) = M\bar{x}(t), \quad (12)$$

where the constant matrix M is a block modal matrix of E_n and determined by means of the extended matrix sign function shown below.

Step 1. Find sign $(\tilde{E}_n)$ using the extended matrix sign function with an adequate ρ, where

$$\tilde{E}_n = (E_n-\rho I_n)(E_n+\rho I_n)^{-1}.$$

Step 2. Find $\mathrm{sign}^+(\tilde{E}_n) = \frac{1}{2}[I_n+\mathrm{sign}(\tilde{E}_n)]$ and $\mathrm{sign}^-(\tilde{E}_n) = \frac{1}{2}[I_n-\mathrm{sign}(\tilde{E}_n)]$.

Step 3. Construct $M = [\mathrm{ind}(\mathrm{sign}^+(\tilde{E}_n))\ \mathrm{ind}(\mathrm{sign}^-(\tilde{E}_n))]$, where ind $(\cdot)$ represents the collection of the linearly independent column vectors of $(\cdot)$.

Substituting (12) back to (11), and multiplying M^{-1} on the left of the equality, one has

$$M^{-1}E_nM\dot{\bar{x}}(t) = M^{-1}A_nM\bar{x}(t)+M^{-1}B_nu(t)$$
$$= M^{-1}\frac{1}{\beta}(I_n-\alpha E_n)M\bar{x}(t)+M^{-1}B_nu(t)$$

$$\Rightarrow \left[\begin{array}{c|c} \bar{E}_1 & O \\ \hline O & \bar{E}_2 \end{array}\right]\dot{\bar{x}}(t)$$

$$= \left[\begin{array}{c|c} \frac{1}{\beta}(I_k-\alpha\bar{E}_1) & O \\ \hline O & \frac{1}{\beta}(I_{n-k}-\alpha\bar{E}_2) \end{array}\right]\bar{x}(t)+\left[\begin{array}{c} \bar{B}_1 \\ \hline \bar{B}_2 \end{array}\right]u(t), \quad (13)$$

where $\bar{x}(t) = [\bar{x}_s'(t), \bar{x}_f'(t)]'$, $M^{-1}E_nM =$ block diagonal $\{\bar{E}_1, \bar{E}_2\}$. $\bar{E}_1$ is invertible with rank $(\bar{E}_1) = \deg\{\det(sE_r-A_r)\} \triangleq k$. $\bar{E}_2$ is a nilpotent matrix with dimension $(n-k)\times(n-k)$, and $[\bar{B}_1', \bar{B}_2']' = M^{-1}B_n$. Notice that since $\det(I_{n-k}-\alpha\bar{E}_2) = 1$, it is invertible. Simplifying (13) by premultiplying the block diagonal $\{\bar{E}_1^{-1}, \beta(I_{n-k}-\alpha\bar{E}_2)^{-1}\}$ on both sides of the equality, one obtains

$$\left[\begin{array}{c|c} I_k & O \\ \hline O & \beta(I_{n-k}-\alpha\bar{E}_2)^{-1}\bar{E}_2 \end{array}\right]\dot{\bar{x}}(t) = \left[\begin{array}{c|c} \frac{1}{\beta}(\bar{E}_1^{-1}-\alpha I_k) & O \\ \hline O & I_{n-k} \end{array}\right]\bar{x}(t) + \left[\begin{array}{c} \bar{E}_1^{-1}\bar{B}_1 \\ \hline \beta(I_{n-k}-\alpha\bar{E}_2)^{-1}\bar{B}_2 \end{array}\right]u(t)$$

$$\Rightarrow \left[\begin{array}{c|c} I_k & O \\ \hline O & \bar{E}_f \end{array}\right]\dot{\bar{x}}(t) = \left[\begin{array}{c|c} \bar{A}_s & O \\ \hline O & I_{n-k} \end{array}\right]\bar{x}(t) + \left[\begin{array}{c} \bar{B}_s \\ \hline \bar{B}_f \end{array}\right]u(t), \qquad (14)$$

where $\bar{E}_f = \beta(I_{n-k}-\alpha\bar{E}_2)^{-1}\bar{E}_2$, $\bar{A}_s = 1/\beta(\bar{E}_1^{-1}-\alpha I_k)$, $\bar{B}_s = \bar{E}_1^{-1}\bar{B}_1$ and $\bar{B}_f = \beta(I_{n-k}-\alpha\bar{E}_2)^{-1}\bar{B}_2$. It is remarkable to note that since

$$\operatorname{rank}(E_r) - \deg\{\det(sE_r - A_r)\} = \operatorname{rank}(\bar{E}_f), \qquad (15)$$

it is much easier to determine the number of the impulsive mode using the above equation relating to (14).

In order to avoid the complexity of statement, we discuss only those kind of singular systems which include at least one impulsive mode. First, assume that the singular system (14) has q impulsive modes, then $\operatorname{rank}(\bar{E}_f) = q$. It should be emphasized that since the nilpotent matrix $\bar{E}_f$, in general, is not in the Jordan block form, it is necessary to solve the eigenvector problem (**17**, **18**) for $\bar{E}_f$. The following proposed method is more convenient for finding the preliminary feedback gain K_f and to prove that K_f can eliminate the impulsive modes.

Let

$$\bar{x}(t) = V\hat{x}(t), \qquad (16)$$

where $\hat{x}(t) = [\hat{x}_s'(t), \hat{x}_f(t)']' = [\bar{x}_s'(t), (U^{-1}\bar{x}_f(t))']'$, and

$$V = \left[\begin{array}{c|c} I_k & O \\ \hline O & U \end{array}\right].$$

U is a modal matrix of E_f with dimension $(n-k)\times(n-k)$ such that $U^{-1}\bar{E}_fU$ is in the Jordan block form. Substituting (16) into (14) and premultiplying it by V^{-1}, we obtain

$$\left[\begin{array}{c|c} I_k & O \\ \hline O & \hat{E}_f \end{array}\right]\dot{\hat{x}}(t) = \left[\begin{array}{c|c} \hat{A}_s & O \\ \hline O & I_{n-k} \end{array}\right]\hat{x}(t) + \left[\begin{array}{c} \hat{B}_s \\ \hline \hat{B}_f \end{array}\right]u(t), \qquad (17)$$

where $\hat{E}_f = U^{-1}\bar{E}_fU$, $\hat{A}_s = \bar{A}_s$, $\hat{B}_s = \bar{B}_s$ and $\hat{B}_f = U^{-1}\bar{B}_f$. Notice that $\hat{E}_f$ is in the

Jordan block form with d blocks of sizes $\mu_1, \mu_2, \ldots, \mu_d$, where $\sum_{i=1}^{d} \mu_i =$ column (row) number of $\hat{E}_f$. Taking the Laplace transformation of the fast subsystem $\hat{E}_f \dot{\hat{x}}_f(t) = \hat{x}_f(t) + \hat{B}_f u(t)$ in (17), one obtains

$$\begin{aligned} \hat{X}_f(s) &= (s\hat{E}_f - I_{n-k})^{-1}(\hat{E}_f \hat{x}_f(0) + \hat{B}_f U(s)) \\ &= -\sum_{i=0}^{l-1} s^i \hat{E}_f^i (\hat{E}_f \hat{x}_f(0) + \hat{B}_f U(s)), \end{aligned} \tag{18}$$

where $\hat{X}_f(s)$ and $U(s)$ denote the Laplace transformations of $\hat{x}_f(t)$ and $u(t)$, respectively. $\hat{x}_f(0)$ denotes the initial value of $\hat{x}_f(t)$, and l is said to be the nilpotency index of $\hat{E}_f$. Taking the inverse Laplace transformation of the above equation, we have the well-known result (7)

$$\hat{x}_f(t) = -\sum_{i=1}^{l-1} \hat{E}_f^i \hat{x}_f(0)\delta^{(i-1)}(t) - \sum_{i=0}^{l-1} \hat{E}_f^i \hat{B}_f u^{(i)}(t), \tag{19}$$

where $\delta(t)$ and $\delta^{(i)}(t)$ denote the delta function and the ith derivative of the delta function, respectively. Apparently, it shows that the impulsive modes of the fast state result from inconsistent initial conditions of the fast state or discontinuous control input (or its derivatives).

Here, we propose a preliminary feedback design method to eliminate the impulsive modes, which is simpler than Cobb's (**8**). The method for determining the preliminary feedback gain $K_f = [k_1, k_2, \ldots, k_{n-k}]_{m \times (n-k)}$, where k_j is of dimension $m \times 1$ for $j = 1, 2, \ldots, (n-k)$, is summarized as follows.

1. If $\mu_i \geqslant 1$, where $1 \leqslant i \leqslant d$, and its corresponding Jordan block is a null matrix, then

$$\begin{aligned} k_{\mu_1+\mu_2+\cdots+\mu_{i-1}+1} &= O_{m\times 1} \\ k_{\mu_1+\mu_2+\cdots+\mu_{i-1}+2} &= O_{m\times 1} \\ &\vdots \\ k_{\mu_1+\mu_2+\cdots+\mu_i} &= O_{m\times 1}. \end{aligned}$$

2. If $\mu_i > 1$, where $1 \leqslant i \leqslant d$, and its corresponding Jordan block is not a null matrix, then

$$\begin{aligned} k_{\mu_1+\mu_2+\cdots+\mu_{i-1}+1} &= \begin{bmatrix} \delta(\hat{b}_{(\mu_1+\mu_2+\cdots+\mu_i)1}) \\ \delta(\hat{b}_{(\mu_1+\mu_2+\cdots+\mu_i)2}) \\ \vdots \\ \delta(\hat{b}_{(\mu_1+\mu_2+\cdots+\mu_i)m}) \end{bmatrix} \\ k_{\mu_1+\mu_2+\cdots+\mu_{i-1}+2} &= O_{m\times 1} \\ &\vdots \\ k_{\mu_1+\mu_2+\cdots+\mu_i} &= O_{m\times 1}, \end{aligned}$$

where

$$\hat{B}_{\mathrm{f}} \triangleq \begin{bmatrix} \hat{b}_{k+1} \\ \hat{b}_{k+2} \\ \vdots \\ \hat{b}_n \end{bmatrix}_{(n-k)\times m}, \quad \hat{b}_i \triangleq [\hat{b}_{i1} \quad \hat{b}_{i2} \quad \dots \quad \hat{b}_{im}]_{1\times m}$$

$$\delta(\hat{b}_{ij}) \triangleq \begin{cases} 0 & \text{if } \hat{b}_{ij} = 0 \\ 1 & \text{if } \hat{b}_{ij} > 0 \quad j = 1, 2, \dots, m. \\ -1 & \text{if } \hat{b}_{ij} < 0 \end{cases}$$

Let

$$\begin{aligned} u(t) &= -K_{\mathrm{f}}\hat{x}_{\mathrm{f}}(t) + v(t) \\ &= -[O_{m\times k}, K_{\mathrm{f}}]\hat{x}(t) + v(t). \end{aligned} \tag{20}$$

Substituting (20) back to (17) yields

$$E_k\dot{\hat{x}}(t) = A_k\hat{x}(t) + B_k v(t), \tag{21}$$

where

$$E_k = \left[\begin{array}{c|c} I_k & O \\ \hline O & \hat{E}_{\mathrm{f}} \end{array}\right], \quad A_k = \left[\begin{array}{c|c} \hat{A}_{\mathrm{s}} & -\hat{B}_{\mathrm{s}}K_{\mathrm{f}} \\ \hline O & I_{n-k} - \hat{B}_{\mathrm{f}}K_{\mathrm{f}} \end{array}\right], \quad B_k = \begin{bmatrix} \hat{B}_{\mathrm{s}} \\ \cdot \\ \hat{B}_{\mathrm{f}} \end{bmatrix}.$$

Lemma 2

The singular system in (21) has the original k finite modes and another $q(=\operatorname{rank}(\bar{E}_{\mathrm{f}}) = \operatorname{rank}(\hat{E}_{\mathrm{f}}))$ finite modes that were induced by applying a linear preliminary feedback control law $u(t)$ in (20) to the system in (17). All these finite modes are guaranteed to be controllable.

Here, we have to emphasize that with Cobb's method for determining the preliminary feedback control in (20), we may need to find an invertible matrix and execute the operations of elementary row and column interchange. Our method, however, is very easy and efficient requiring no computation.

Now, we want to decompose the singular system into a reduced-order regular system with $k+q$ controllable finite modes and a nondynamic equation with $n-k-q$ infinite nondynamic ones. It can be accomplished by using previously outlined steps once again. First, we transform the regular form into a standard one by premultiplying (21) by $(\gamma E_k + \eta A_k)^{-1}$, where γ and η are arbitrary scalars such that $(\gamma E_k + \eta A_k)$ is invertible. Therefore, we obtain

$$(\gamma E_k + \eta A_k)^{-1}E_k\dot{\hat{x}}(t) = (\gamma E_k + \eta A_k)^{-1}A_k\hat{x}(t) + (\gamma E_k + \eta A_k)^{-1}B_k v(t). \tag{22}$$

Let

$$\hat{x}(t) = \tilde{M}\tilde{x}(t), \tag{23}$$

where the constant matrix $\tilde{M}$ is determined using the extended matrix sign function.

The procedure is the same as in the previous illustration for finding M, except that it operates on $(\gamma E_k+\eta A_k)^{-1}E_k$. Substituting (23) into (22), and premultiplying it by $\tilde{M}^{-1}$, one gets

$$\begin{aligned}
\tilde{M}^{-1}(\gamma E_k+\eta A_k)^{-1}E_k\tilde{M}\dot{\tilde{x}}(t)
&= \tilde{M}^{-1}(\gamma E_k+\eta A_k)^{-1}A_k\tilde{M}\tilde{x}(t)+\tilde{M}^{-1}(\gamma E_k+\eta A_k)^{-1}B_k r(t)\\
&= \tilde{M}^{-1}\frac{1}{\eta}[I_n-\gamma(\gamma E_k+\eta A_k)^{-1}E_k]\tilde{M}\tilde{x}(t)+\tilde{M}^{-1}(\gamma E_k+\eta A_k)^{-1}B_k r(t)\\
&= \frac{1}{\eta}[I_n-\gamma\tilde{M}^{-1}(\gamma E_k+\eta A_k)^{-1}E_k\tilde{M}]\tilde{x}(t)+\tilde{M}^{-1}(\gamma E_k+\eta A_k)^{-1}B_k r(t)
\end{aligned}$$

$$\Rightarrow \left[\begin{array}{c|c} \bar{E}_{sk} & O \\ \hline O & \bar{E}_{fk} \end{array}\right]\dot{\tilde{x}}(t) = \left[\begin{array}{c|c} \frac{1}{\eta}(I_{k+q}-\gamma\bar{E}_{sk}) & O \\ \hline O & \frac{1}{\eta}(I_{n-k-q}-\gamma\bar{E}_{fk}) \end{array}\right]\tilde{x}(t)+\begin{bmatrix} \bar{B}_{sk} \\ \bar{B}_{fk} \end{bmatrix} r(t)$$

$$\Rightarrow \left[\begin{array}{c|c} \bar{E}_{sk} & O \\ \hline O & O_{(n-q-k)} \end{array}\right]\dot{\tilde{x}}(t) = \left[\begin{array}{c|c} \frac{1}{\eta}(I_{k+q}-\gamma\bar{E}_{sk}) & O \\ \hline O & \frac{1}{\eta}I_{n-k-q} \end{array}\right]\tilde{x}(t)+\begin{bmatrix} \bar{B}_{sk} \\ \bar{B}_{fk} \end{bmatrix} r(t). \quad (24)$$

where $\tilde{x}(t) = [\tilde{x}_s'(t), \tilde{x}_f'(t)]'$, $\tilde{M}^{-1}(\gamma E_k+\eta A_k)^{-1}E_k\tilde{M}$ = block diagonal $\{\bar{E}_{sk}, \bar{E}_{fk}\}$ = block diagonal $\{\bar{E}_{sk}, O_{(n-k-q)}\}$. $\bar{E}_{sk}$ is invertible with rank $(\bar{E}_{sk})$ = deg $\{\det(sE_k-A_k)\} = (q+k)$. $\bar{E}_{fk}$ is a null matrix with dimension $(n-k-q)\times(n-k-q)$, and $[\bar{B}_{sk}', \bar{B}_{fk}']' = \tilde{M}^{-1}(\gamma E_k+\eta A_k)^{-1}B_k$.

IV. Model Conversion

It is clear to see that the above equation in (24) can be decomposed into a reduced-order regular system and a nondynamic equation as follows:

$$\dot{\tilde{x}}_s(t) = \frac{1}{\eta}(\bar{E}_{sk}^{-1}-\gamma I_{k+q})\tilde{x}_s(t)+\bar{E}_{sk}^{-1}\bar{B}_{sk}r(t) \triangleq \tilde{A}_{sk}\tilde{x}_s(t)+\tilde{B}_{sk}r(t) \quad (25)$$

$$\tilde{x}_f(t) = -\eta\bar{B}_{fk}r(t). \quad (26)$$

If $r(t)$ is a piecewise-constant input, i.e. $r(t) = r(kT)$ for $kT \leqslant t < kT+T$, where T is a sampling period, then we have the discrete-time system corresponding to (25) and (26) in the following:

$$\bar{x}_{ds}(kT+T) = \tilde{G}_s \bar{x}_{ds}(kT) + \tilde{H}_s r(kT) \tag{27}$$

$$\bar{x}_{df}(kT) = -\tilde{H}_f r(kT), \tag{28}$$

where $\tilde{G}_s = \exp(\tilde{A}_{sk}T)$, $\tilde{H}_s = (\tilde{G}_s - I_{q+k})\tilde{A}_{sk}^{-1}\bar{B}_{sk}$ and $\tilde{H}_f = \eta \bar{B}_{fk}$.

In general, the matrices $\tilde{G}_s$ and $\tilde{H}_s$ can be determined exactly using the eigenvalue eigenvector approach (**19**). However, approximations are required for obtaining $\tilde{G}_s$ and $\tilde{H}_s$ matrices when matrix $\tilde{A}_{sk}$ is singular. There are many methods (**19**) available to evaluate approximately $\tilde{G}_s$ and $\tilde{H}_s$; especially, using Pade's approximation method (**19–20**), which is more popular. Some of the approximations obtained using the Pade's approximation method are listed below:

$$\tilde{G}_s \approx (I_{q+k} - \tfrac{1}{2}\tilde{A}_{sk}T)^{-1}(I_{q+k} + \tfrac{1}{2}\tilde{A}_{sk}T) \triangleq \tilde{G}_{s3} \tag{29}$$

$$\approx [I_{q+k} - \tfrac{1}{2}\tilde{A}_{sk}T + \tfrac{1}{12}(\tilde{A}_{sk}T)^2]^{-1}[I_{q+k} + \tfrac{1}{2}\tilde{A}_{sk}T + \tfrac{1}{12}(\tilde{A}_{sk}T)^2] \triangleq \tilde{G}_{s5} \tag{30}$$

and

$$\tilde{H}_s \approx T(I_{q+k} - \tfrac{1}{2}\tilde{A}_{sk}T)^{-1}\bar{B}_{sk} \triangleq \tilde{H}_{s3} \tag{31}$$

$$\approx T[I_{q+k} - \tfrac{1}{2}\tilde{A}_{sk}T + \tfrac{1}{12}(\tilde{A}_{sk}T)^2]^{-1}\bar{B}_{sk} \triangleq \tilde{H}_{s5}. \tag{32}$$

Combining (27) with (28) yields

$$\begin{bmatrix} I_{q+k} & | & O \\ & + & \\ O & | & O_{n-q-k} \end{bmatrix} \begin{bmatrix} \bar{x}_{ds}(kT+T) \\ \\ \bar{x}_{df}(kT+T) \end{bmatrix} = \begin{bmatrix} \tilde{G}_s & | & O \\ & + & \\ O & | & I_{n-q-k} \end{bmatrix} \begin{bmatrix} \bar{x}_{ds}(kT) \\ \\ \bar{x}_{df}(kT) \end{bmatrix} + \begin{bmatrix} \tilde{H}_s \\ \\ \tilde{H}_f \end{bmatrix} r(kT). \tag{33}$$

We transform (33) back to that of the appropriate discrete-time system coordinate, which is corresponding to the case of the continuous-time system $E_r\dot{x}_d(t) = (A_r - B_rK_p)x_d(t) + B_rr(t)$ where $K_p = [O_{m\times k}, K_f](MV)^{-1}$, as follows:

$$(MV\tilde{M}) \begin{bmatrix} I_{q+k} & | & O \\ & + & \\ O & | & O_{n-q-k} \end{bmatrix} (MV\tilde{M})^{-1} x_d(kT+T) = (MV\tilde{M}) \begin{bmatrix} \tilde{G}_s & | & O \\ & + & \\ O & | & I_{n-q-k} \end{bmatrix} (MV\tilde{M})^{-1} x_d(kT) + (MV\tilde{M}) \begin{bmatrix} \tilde{H}_s \\ \\ \tilde{H}_f \end{bmatrix} r(kT). \tag{34}$$

V. Digital Redesign

Consider the slow subsystem and the fast subsystem described by (25) and (26), respectively. Also, let the optimal control law obtained by using the method of Shieh *et al.* (**21**) for the slow subsystem be

$$u(t) = -K_{cs}\tilde{x}_s(t) + E_{cs}r(t). \tag{35}$$

Thus the closed-loop system becomes

$$\dot{\tilde{x}}_s(t) = (\tilde{A}_{sk} - \tilde{B}_{sk}K_{cs})\tilde{x}_s(t) + \tilde{B}_{sk}E_{cs}r(t) \tag{36}$$

$$\tilde{x}_f(t) = \eta\bar{B}_{fk}K_{cs}\tilde{x}_s(t) - \eta\bar{B}_{fk}E_{cs}r(t) \tag{37}$$

and its eigenvalues are located on or within the hatched region of Fig. 1.

Assuming $r(t) = r(kT)$ for $kT \leqslant t < kT+T$, we have the respective discrete-time models of (36) and (37) as follows:

$$\tilde{x}_{ds}(kT+T) = G_s\tilde{x}_{ds}(kT) + H_s r(kT) \tag{38}$$

$$\tilde{x}_{df}(kT) = \eta\bar{B}_{fk}K_{cs}\tilde{x}_{ds}(kT) - \eta\bar{B}_{fk}E_{cs}r(kT), \tag{39}$$

where $G_s = \exp((\tilde{A}_{sk} - \tilde{B}_{sk}K_{cs})T)$, $H_s = (G_s - I_{q\cdot k})(\tilde{A}_{sk} - \tilde{B}_s K_{cs})^{-1}\tilde{B}_{sk}E_{cs}$. Its eigenvalues are located on or within the hatched region of Fig. 2.

Suppose a digital model which approximates the slow subsystem in (**25**) is represented by

$$\dot{\tilde{x}}_d(t) = \tilde{A}_{sk}\tilde{x}_d(t) + \tilde{B}_{sk}r(kT). \tag{40}$$

Then the equivalent discrete-time model can be written as

$$\tilde{x}_d(kT+T) = \tilde{G}_d\tilde{x}_d(kT) + \tilde{H}_d r(kT). \tag{41}$$

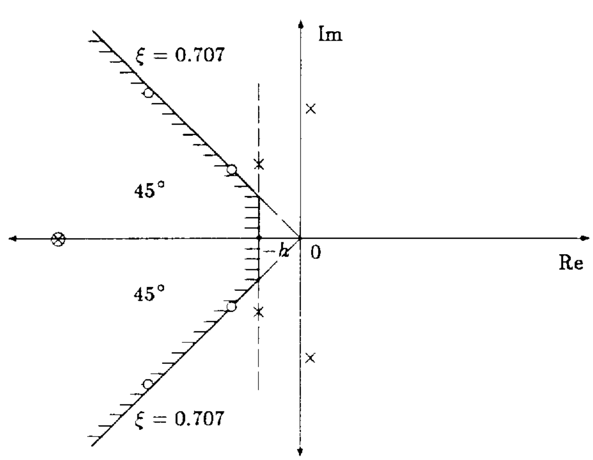

FIG. 1. Region of interest in the continuous-time s-plane. × Open-loop poles before design. ○ Closed-loop poles after design.

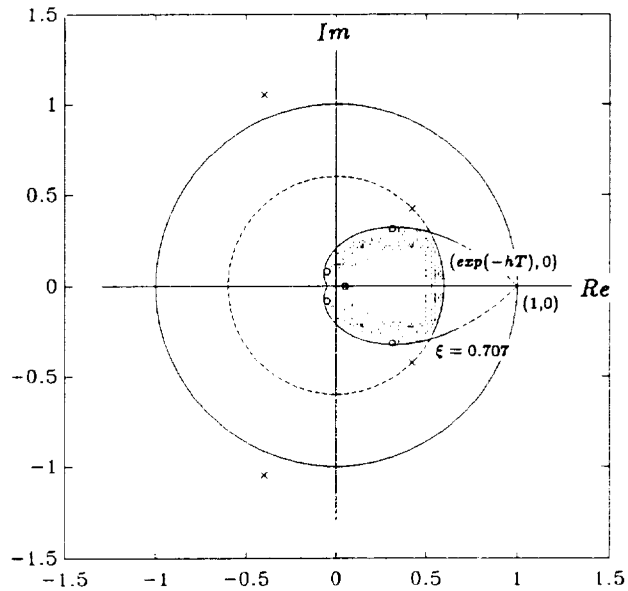

FIG. 2. Region of interest in the z-plane. × Poles before design. ○ Poles after design.

where $\tilde{G}_d = \exp(\tilde{A}_{sk}T)$, $\tilde{H}_d = (\tilde{G}_d - I_{q+k})\tilde{A}_{sk}^{-1}\tilde{B}_{sk}$. Let the digital control law for the discrete-time model in (41) be

$$u(kT) = -K_d\tilde{x}_d(kT) + E_d r(kT). \tag{42}$$

Then we have the designed closed-loop subsystem as

$$\tilde{x}_d(kT+T) = (\tilde{G}_d - \tilde{H}_d K_d)\tilde{x}_d(kT) + \tilde{H}_d E_d r(kT). \tag{43}$$

To match $\tilde{x}_{ds}(kT) \approx \tilde{x}_d(kT)$ with a sufficiently small sampling period T and the same inputs, as well as the initial conditions, the explicit feedback gain K_d and forward gain E_d have already been solved by Tsai *et al.* (**2**) as follows.

First, G_s can be approximated by using the Pade's approximation method as

$$G_s = \exp((\tilde{A}_{sk} - \tilde{B}_{sk}K_{cs})T) \approx [I_{q+k} - \tfrac{1}{2}(\tilde{A}_{sk} - \tilde{B}_{sk}K_{cs})T]^{-1}[I_{q+k} + \tfrac{1}{2}(\tilde{A}_{sk} - \tilde{B}_{sk}K_{cs})T]. \tag{44}$$

Next, based on the fact that

$$(A+BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}, \tag{45}$$

we can find the important results from the following derivations:

$$
\begin{aligned}
G_s &\approx [I_{q+k} - \tfrac{1}{2}(\tilde{A}_{sk} - \tilde{B}_{sk}K_{cs})T]^{-1}[I_{q+k} + \tfrac{1}{2}(\tilde{A}_{sk} - \tilde{B}_{sk}K_{cs})T] \\
&= [(I_{q+k} - \tfrac{1}{2}\tilde{A}_{sk}T) + \tilde{B}_{sk}T\tfrac{1}{2}I_m K_{cs}]^{-1}[I_{q+k} + \tfrac{1}{2}(\tilde{A}_{sk} - \tilde{B}_{sk}K_{cs})T] \\
&= \{(I_{q+k} - \tfrac{1}{2}\tilde{A}_{sk}T)^{-1} - (I_{q+k} - \tfrac{1}{2}\tilde{A}_{sk}T)^{-1}\tilde{B}_{sk}T[2I_m + K_{cs}(I_{q+k} \\
&\quad - \tfrac{1}{2}\tilde{A}_{sk}T)^{-1}\tilde{B}_{sk}T]^{-1}K_{cs}(I_{q+k} - \tfrac{1}{2}\tilde{A}_{sk}T)^{-1}\}[I_{q+k} + \tfrac{1}{2}(\tilde{A}_{sk} - \tilde{B}_{sk}K_{cs})T] \\
&= (I_{q+k} - \tfrac{1}{2}\tilde{A}_{sk}T)^{-1}(I_{q+k} + \tfrac{1}{2}\tilde{A}_{sk}T) - (I_{q+k} - \tfrac{1}{2}\tilde{A}_{sk}T)^{-1}\tilde{B}_{sk}T[2I_m \\
&\quad + K_{cs}(I_{q+k} - \tfrac{1}{2}\tilde{A}_{sk}T)^{-1}\tilde{B}_{sk}T]^{-1}K_{cs}(I_{q+k} - \tfrac{1}{2}\tilde{A}_{sk}T)^{-1}(I_{q+k} + \tfrac{1}{2}\tilde{A}_{sk}T) \\
&\quad - \tfrac{1}{2}(I_{q+k} - \tfrac{1}{2}\tilde{A}_{sk}T)^{-1}\tilde{B}_{sk}TK_{cs} + \tfrac{1}{2}(I_{q+k} - \tfrac{1}{2}\tilde{A}_{sk}T)^{-1}\tilde{B}_{sk}T[2I_m \\
&\quad + K_{cs}(I_{q+k} - \tfrac{1}{2}\tilde{A}_{sk}T)^{-1}\tilde{B}_{sk}T]^{-1}K_{cs}(I_{q+k} - \tfrac{1}{2}\tilde{A}_{sk}T)^{-1}\tilde{B}_{sk}TK_{cs} \\
&\approx \tilde{G}_d - \tilde{H}_d(2I_m + K_{cs}\tilde{H}_d)^{-1}K_{cs}\tilde{G}_d - \tfrac{1}{2}\tilde{H}_dK_{cs} + \tfrac{1}{2}\tilde{H}_d(2I_m + K_{cs}\tilde{H}_d)^{-1}K_{cs}\tilde{H}_dK_{cs} \\
&= \tilde{G}_d - \tilde{H}_d[\tfrac{1}{2}(I_m + \tfrac{1}{2}K_{cs}\tilde{H}_d)^{-1}K_{cs}(I_{q+k} + \tilde{G}_d)] \\
&= \tilde{G}_d - \tilde{H}_dK_d \qquad (46)
\end{aligned}
$$

and

$$
\begin{aligned}
H_s &= (G_s - I_{q+k})(\tilde{A}_{sk} - \tilde{B}_{sk}K_{cs})^{-1}\tilde{B}_{sk}E_{cs} \\
&\approx [I_{q+k} - \tfrac{1}{2}(\tilde{A}_{sk} - \tilde{B}_{sk}K_{cs})T]^{-1}\tilde{B}_{sk}TE_{cs} \\
&\approx \tilde{H}_d(I_m + \tfrac{1}{2}K_{cs}\tilde{H}_d)^{-1}E_{cs} \\
&= \tilde{H}_dE_d. \qquad (47)
\end{aligned}
$$

Hence

$$K_d = \tfrac{1}{2}(I_m + \tfrac{1}{2}K_{cs}\tilde{H}_d)^{-1}K_{cs}(I_{q+k} + \tilde{G}_d) \qquad (48)$$

$$E_d = (I_m + \tfrac{1}{2}K_{cs}\tilde{H}_d)^{-1}E_{cs}. \qquad (49)$$

Finally, the state feedback control law $u(kT)$ can be derived through some adequate coordinate transformation and the necessary definition,

$$
\begin{aligned}
x_d(kT) &\triangleq M\begin{bmatrix}\bar{x}_d(kT)\\ \bar{x}_f(kT)\end{bmatrix} \triangleq MV\begin{bmatrix}\hat{x}_d(kT)\\ \hat{x}_f(kT)\end{bmatrix} \triangleq MV\tilde{M}\begin{bmatrix}\tilde{x}_d(kT)\\ \tilde{x}_f(kT)\end{bmatrix} \\
&\approx MV\tilde{M}\begin{bmatrix}\tilde{x}_{ds}(kT)\\ \tilde{x}_{df}(kT)\end{bmatrix} = x(kT). \qquad (50)
\end{aligned}
$$

By the above definitions, one gets

$$
\begin{aligned}
u(t) &= -K_f\hat{x}_f(t) + v(t) \\
&= -[O_{m\times k}, \quad K_f]\begin{bmatrix}\hat{x}_s(t)\\ \hat{x}_f(t)\end{bmatrix} + v(t) \\
&= -[O_{m\times k}, K_f](MV)^{-1}x(t) + v(t) \\
&\triangleq -K_px(t) + v(t) \qquad (51)
\end{aligned}
$$

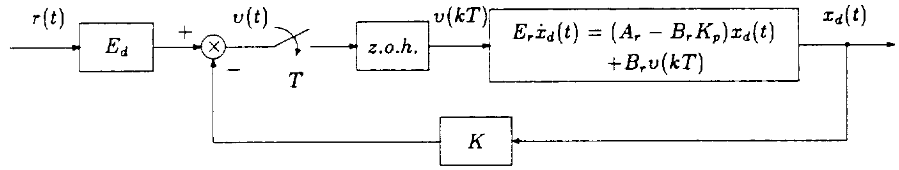

FIG. 3. Digital redesign system.

and

$$\begin{aligned} v(kT) &= -K_{\mathrm{d}}\tilde{x}_{\mathrm{d}}(kT) + E_{\mathrm{d}}r(kT) \\ &= -[K_{\mathrm{d}}, O_{m\times(n-q-k)}]\begin{bmatrix}\tilde{x}_{\mathrm{d}}(kT) \\ \tilde{x}_{\mathrm{f}}(kT)\end{bmatrix} + E_{\mathrm{d}}r(kT) \\ &= -[K_{\mathrm{d}}, O_{m\times(n-q-k)}](MV\tilde{M})^{-1}x_{\mathrm{d}}(kT) + E_{\mathrm{d}}r(kT) \\ &\triangleq -Kx_{\mathrm{d}}(kT) + E_{\mathrm{d}}r(kT), \end{aligned} \tag{52}$$

where

$$K_p = [O_{m\times k}, K_{\mathrm{f}}](MV)^{-1}, \quad K = [K_{\mathrm{d}}, O_{m\times(n-q-k)}](MV\tilde{M})^{-1}.$$

The redesigned digital system is shown in Fig. 3. It is noted that when the discrete-time state $x_{\mathrm{d}}(kT)$ is not accessible, the ideal state reconstructor methods **(22–23)** can be applied to reconstruct the exact discrete-time state $x_{\mathrm{d}}(kT)$ using the input data and fast-rate output data of the original continuous-time system without establishing an observer.

VI. Illustration Example

Consider a linear continuous-time singular system described in (1) with

$$E_r = \begin{bmatrix} 1 & 2 & 1 & 1 & -3 & -2 \\ 0 & 2 & 2 & 1 & -3 & -3 \\ 1 & 2 & 1 & 1 & -3 & -2 \\ 1 & 2 & 1 & 3 & -5 & -4 \\ 0 & 2 & 1 & 1 & -2 & -2 \\ 1 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}, \quad A_r = I_6, \quad B_r' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{bmatrix}.$$

Since $0E_r + A_r = A_r = I_6$, by definition of the standard form, $\{E_r, A_r\}$ is in the standard form. In other words, if we take $\alpha = 0$ and $\beta = 1$, then $E_n = E_r$, $A_n = A_r$ and $B_n = B_r$. Because E_n is singular, i.e. E_n includes some zero eigenvalues, utilizing the bilinear transform to find the similarity transformation matrix M of E_n is necessary. Taking $\rho = 0.5$ and using the algorithm described in Section III, one has

$$\tilde{E} = (E_n - \rho I_6)(E_n + \rho I_6)^{-1} = \begin{bmatrix} 0.3333 & 1.6 & -2.4 & 0.16 & 0.9067 & 2.24 \\ 0 & 0.6 & 1.6 & 0.16 & -1.76 & -1.76 \\ 1.3333 & 1.6 & -3.4 & 0.16 & 0.9067 & 2.24 \\ 1.3333 & 1.6 & -2.4 & 0.76 & -0.6933 & 0.64 \\ 0 & 1.6 & -2.4 & 0.16 & 1.24 & 2.24 \\ 1.3333 & 0 & 0 & 0 & -1.3333 & -1 \end{bmatrix}$$

$$\operatorname{sign}(\tilde{E}) = \begin{bmatrix} 1 & 2 & 2 & 0 & -4 & 2 \\ 0 & 1 & 2 & 0 & -2 & -2 \\ 2 & 2 & 1 & 0 & -4 & 2 \\ 2 & 2 & 2 & 1 & -6 & -4 \\ 0 & 2 & 2 & 0 & -3 & -2 \\ 2 & 0 & 0 & 0 & -2 & -1 \end{bmatrix},$$

$$\operatorname{sign}^{-1}(\tilde{E}) = \begin{bmatrix} 1 & 1 & 1 & 0 & -2 & -1 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 0 & -2 & -1 \\ 1 & 1 & 1 & 1 & -3 & -2 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ 1 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\operatorname{sign}\ (E) = \begin{bmatrix} 0 & -1 & -1 & 0 & 2 & 1 \\ 0 & 0 & -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 & 2 & 1 \\ -1 & -1 & -1 & 0 & 3 & 2 \\ 0 & -1 & -1 & 0 & 2 & 1 \\ -1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$M = [\operatorname{ind}(\operatorname{sign}^{-1}(E))\,\operatorname{ind}(\operatorname{sign}\ (\tilde{E}))] = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & -1 & -1 & 0 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & -1 & -1 \\ 1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}.$$

From (13), we obtain

$$M^{-1}E_nM = \begin{bmatrix} \bar{E}_1 & | & O \\ - & + & - \\ O & | & \bar{E}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & | & 0 & 0 & 0 \\ 0 & 2 & 1 & | & 0 & 0 & 0 \\ 0 & 0 & 2 & | & 0 & 0 & 0 \\ - & - & - & + & - & - & - \\ 0 & 0 & 0 & | & 0 & 0 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 1 \\ 0 & 0 & 0 & | & 0 & 0 & 0 \end{bmatrix}$$

$$M^{-1}A_nM = \begin{bmatrix} I_3 & | & O \\ - & + & - \\ O & | & I_3 \end{bmatrix}$$

$$M^{-1}B_n = [\bar{B}'_1 \,|\, \bar{B}'_2]' = \begin{bmatrix} 1 & 1 & 1 & | & 2 & 0 & 1 \\ 0 & -1 & 1 & | & 0 & 0 & -1 \end{bmatrix}'.$$

Based on (14) and the fact that $\bar{E}_f$ is in the Jordan form, one has

$$V = I_6, \quad \hat{E}_f = \bar{E}_f\bar{E}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \hat{A}_s = \bar{A}_s = \frac{1}{\beta}\bar{E}_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & -0.25 \\ 0 & 0 & 0.5 \end{bmatrix}$$

$$\hat{B}_s = \bar{B}_s = \bar{E}_1^{-1}\bar{B}_1 = \begin{bmatrix} 1 & 0 \\ 0.25 & -0.75 \\ 0.5 & 0.5 \end{bmatrix}, \quad \hat{B}_f = \bar{B}_f = \bar{B}_2 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

Since rank $(\bar{E}_f) = q = 1$, the singular system has one impulsive mode by means of (15) associated with (14).

Now, we compute the preliminary feedback gain K_f for eliminating the impulsive mode of the singular system. As $\mu_1 = 1$, $k_1 = O_{2\times 1}$. In addition, since $u_2 = 2$ and its corresponding Jordan block is not a null matrix, one has

$$k_{\mu_1+1} = k_2 = \begin{bmatrix} \delta(b_{(\mu_1+\mu_2)1}) \\ \delta(b_{(\mu_1+\mu_2)2}) \end{bmatrix} = \begin{bmatrix} \delta(b_{31}) \\ \delta(b_{32}) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$k_{\mu_1+\mu_2} = k_3 = O_{2\times 1}.$$

Thus we get the preliminary feedback gain

$$K_f = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

and the control law

$$u(t) = -[O_{2\times 3}, K_f]\hat{x}(t) + v(t) = -\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}\hat{x}(t) + v(t).$$

Computing the closed-loop singular system with respect to the preliminary feedback gain, yields

$$E_k = \left[\begin{array}{c|c} I_3 & O \\ \hline O & \hat{E}_f \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$A_k = \left[\begin{array}{c|c} \hat{A}_s & -\hat{B}_s K_f \\ \hline O & I_3 - \hat{B}_f K_f \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0.5 & -0.25 & 0 & -1 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 \end{array}\right]$$

$$B_k = \begin{bmatrix} \hat{B}_s \\ \hat{B}_f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0.25 & -0.75 \\ 0.5 & 0.5 \\ 2 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

We transform the regular form $\{E_k, A_k\}$ into a standard one once again with $\gamma = 2$ and $\eta = -1$, and use the extended matrix sign function to find a similarity transformation matrix $\tilde{M}$ of E_k

$$\tilde{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Let $\hat{x}(t) = \tilde{M}\tilde{x}(t)$ and compute (24), then one has

$$\tilde{M}^{-1}(\gamma E_k+\eta A_k)^{-1}E_k\tilde{M}=\left[\begin{array}{c|c}\bar{E}_{sk} & O\\ \hline O & O_2\end{array}\right]$$

$$=\left[\begin{array}{cccc|cc}1 & 0 & 0 & -0.3333 & 0 & 0\\ 0 & 0.6667 & -0.1111 & -0.2222 & 0 & 0\\ 0 & 0 & 0.6667 & 0 & 0 & 0\\ 0 & 0 & 0 & 0.6667 & 0 & 0\\ \hline 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{array}\right]$$

$$\frac{1}{\eta}[I_6-\gamma\tilde{M}^{-1}(\gamma E_k+\eta A_k)^{-1}E_k\tilde{M}]=\left[\begin{array}{c|c}\frac{1}{\eta}(I_4-\gamma\bar{E}_{sk}) & O\\ \hline O & \frac{1}{\eta}I_2\end{array}\right]$$

$$=\left[\begin{array}{cccc|cc}1 & 0 & 0 & -0.6667 & 0 & 0\\ 0 & 0.3333 & -0.2222 & -0.4444 & 0 & 0\\ 0 & 0 & 0.3333 & 0 & 0 & 0\\ 0 & 0 & 0 & 0.3333 & 0 & 0\\ \hline 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & -1\end{array}\right]$$

$$\tilde{M}^{-1}(\gamma E_k+\eta A_k)^{-1}B_k=\begin{bmatrix}\bar{B}_{sk}\\ \\ \bar{B}_{fk}\end{bmatrix}=\begin{bmatrix}0.3333 & 0.6667\\ -0.3333 & -0.1111\\ 0.3333 & 0.3333\\ 0.3333 & -0.3333\\ \\ -1 & -1\\ 0.5 & -0.5\end{bmatrix}.$$

Therefore, we get the reduced-order regular subsystem and the nondynamic subsystem in (25) and (26), respectively,

$$\dot{\tilde{x}}_s(t) = \frac{1}{\eta}(\bar{E}_{sk}^{-1} - \gamma I_4)\tilde{x}_s(t) + \bar{E}_{sk}^{-1}\bar{B}_{sk}r(t)$$

$$= \tilde{A}_{sk}\tilde{x}_s(t) + \tilde{B}_{sk}r(t)$$

$$= \begin{bmatrix} 1 & 0 & 0 & -0.5 \\ 0 & 0.5 & -0.25 & -0.5 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix} \tilde{x}_s(t) + \begin{bmatrix} 0.5 & 0.5 \\ -0.25 & -0.25 \\ 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} r(t)$$

and

$$\tilde{x}_f(t) = -\eta\bar{B}_{fk}r(t) = \begin{bmatrix} -1 & -1 \\ 0.5 & -0.5 \end{bmatrix} r(t).$$

(i) *Model conversion*

The discrete-time model corresponding to (25) and (26) is in (27) and (28), respectively, where $T = 0.1$ s.

$$\tilde{G}_s = \exp(\tilde{A}_{sk}T) = \begin{bmatrix} 1.1052 & 0 & 0 & -0.0539 \\ 0 & 1.0513 & -0.0263 & -0.0526 \\ 0 & 0 & 1.0513 & 0 \\ 0 & 0 & 0 & 1.0513 \end{bmatrix}$$

$$\tilde{H}_s = (\tilde{G}_s - I_4)\tilde{A}_{sk}^{-1}\tilde{B}_{sk} = \begin{bmatrix} 0.0513 & 0.0539 \\ -0.0276 & -0.0250 \\ 0.0513 & 0.0513 \\ 0.0513 & -0.0513 \end{bmatrix}, \quad \tilde{H}_f = \eta\bar{B}_{fk} = \begin{bmatrix} 1 & 1 \\ -0.5 & 0.5 \end{bmatrix}.$$

Thus we find the discrete-time singular system corresponding to the continuous-time singular system $E_r\dot{x}_d(t) = (A_r - B_rK_p)x_d(t) + B_rr(kT)$ from (34) as follows:

$$(MV\tilde{M})\left[\begin{array}{c|c} I_4 & O \\ \hline O & O_2 \end{array}\right](MV\tilde{M})^{-1} = \begin{bmatrix} 1 & 1 & -0.5 & 0 & -0.5 & 0.5 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -0.5 & 0 & -0.5 & 0.5 \\ 1 & 1 & -1.5 & 1 & -0.5 & 0.5 \\ 0 & 1 & -0.5 & 0 & 0.5 & 0.5 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$(MV\tilde{M})\begin{bmatrix} \tilde{G}_s & | & O \\ & + & \\ O & | & I_2 \end{bmatrix}(MV\tilde{M})^{-1}$$

$$= \begin{bmatrix} 1.1052 & 0.0513 & -0.1321 & -0.0263 & 0.0532 & 0.1584 \\ 0 & 1.0513 & -0.0536 & -0.0263 & 0.0788 & 0.0788 \\ 0.1052 & 0.0513 & 0.8679 & -0.0263 & 0.0532 & 0.1584 \\ 0.1052 & 0.0513 & -0.1834 & 1.0250 & 0.0532 & 0.1584 \\ 0 & 0.0513 & -0.0782 & -0.0263 & 1.1045 & 0.1045 \\ 0.1052 & 0 & -0.1052 & 0 & 0 & 1.1052 \end{bmatrix}$$

$$(MV\tilde{M})\begin{bmatrix} \tilde{H}_s \\ \tilde{H}_f \end{bmatrix} = \begin{bmatrix} 0.4468 & -0.3942 \\ -0.0788 & 0.0263 \\ -0.5532 & -1.3942 \\ -0.5532 & -1.2916 \\ 0.3955 & -0.4481 \\ -1 & -0.8948 \end{bmatrix}.$$

The continuous-time singular system has four finite eigenvalues, $\{1, 0.5, 0.5, 0.5\}$, and two infinite nondynamic eigenvalues. Its corresponding discrete-time model has four finite eigenvalues, $\{1.1052, 1.0513, 1.0513, 1.0513\}$, and two infinite nondynamic eigenvalues.

(ii) *Digital redesign*

Set the desired control law for the slow subsystem as described in (35) where

$$K_{cs} = \begin{bmatrix} 7.9819 & -23.9457 & -13.8190 & 5.8507 \\ 28.3801 & -1.1403 & -21.8009 & -6.8643 \end{bmatrix}, \quad E_{cs} = I_4.$$

The eigenvalues of the closed-loop continuous-time singular system $E_r\dot{x}(t) = (A_r - B_rK)x(t) + B_rr(t)$, denoted by $\sigma(sE_r - A_r + B_rK)$ where $K = \{[O_{2\times 3}, K_f] + [K_{cs}, O_{2\times 2}]M^{-1}\}(MV)^{-1}$, include two infinite nondynamic eigenvalues and four finite eigenvalues $\{-3, -2.5, -2.5, -2.5\}$ which lie within the specified region with $h = 1$ in Fig. 1. Therefore, one has the equivalent discrete-time model in (41) with

$$\tilde{G}_d = \exp(\tilde{A}_{sk}T) = \begin{bmatrix} 1.1052 & 0 & 0 & -0.0539 \\ 0 & 1.0513 & -0.0263 & -0.0526 \\ 0 & 0 & 1.0513 & 0 \\ 0 & 0 & 0 & 1.0513 \end{bmatrix}$$

$$\tilde{H}_d = (\tilde{G}_d - I_4)\tilde{A}_{sk}^{-1}\tilde{B}_{sk} = \begin{bmatrix} 0.0513 & 0.0539 \\ -0.0276 & -0.0250 \\ 0.0513 & 0.0513 \\ 0.0513 & -0.0513 \end{bmatrix}.$$

Hence K_d and E_d concerning the digital control law in (42) are

$$K_d = \tfrac{1}{2}(I_2+\tfrac{1}{2}K_{cs}\tilde{H}_d)^{-1}K_{cs}(I_4+\tilde{G}_d) = \begin{bmatrix} 6.1532 & -18.4535 & -10.2956 & 4.8638 \\ 21.3588 & -0.7265 & -15.9422 & -5.5982 \end{bmatrix}$$

$$E_d = (I_2+\tfrac{1}{2}K_{cs}\tilde{H}_d)^{-1}E_{cs} = \begin{bmatrix} 0.7516 & -0.0054 \\ -0.0045 & -0.7163 \end{bmatrix}.$$

Thus the state feedback gains K_p and K for continuous-time singular system and the redesigned sampled-data system, respectively, are

$$K_p = [O_{2\times 3}, K_f](MV)^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$K = [K_d, O_{2\times 2}](MV\tilde{M})^{-1} = \begin{bmatrix} 6.1532 & -18.4535 & -13.5897 & -10.2956 & 17.7322 & 23.8853 \\ 21.3588 & -0.7265 & -6.3246 & -15.9422 & 0.9081 & 22.2668 \end{bmatrix}.$$

The eigenvalues of the equivalent discrete-time singular system

$$(MV\tilde{M})\left[\begin{array}{c|c} I_4 & O \\ \hline O & O_2 \end{array}\right](MV\tilde{M})^{-1}x_d(kT+T)$$

$$= (MV\tilde{M})\left\{\left[\begin{array}{c|c} \tilde{G}_s & O \\ \hline O & I_2 \end{array}\right](MV\tilde{M})^{-1} - \begin{bmatrix} \tilde{H}_s \\ \tilde{H}_f \end{bmatrix} K\right\}x_d(kT)$$

$$+ (MV\tilde{M})\begin{bmatrix} \tilde{H}_s \\ \tilde{H}_f \end{bmatrix} E_d r(kT)$$

include two infinite nondynamic eigenvalues and four finite eigenvalues $\{0.7408, 0.7788, 0.7788, 0.7788\}$ which lie within the specified region with $h = 1$ in Fig. 2. The simulation results with respect to controls are shown in Figs 4 and 5, respectively.

VII. Conclusions

The model conversion and digital redesign problems for a regular system have been extended to the case of a singular system which is controllable at finite and impulsive modes. If the singular system does not have any impulsive mode, then we can easily utilize the good characteristics of the standard pair and apply the extended matrix sign function to decompose the singular system into a reduced-

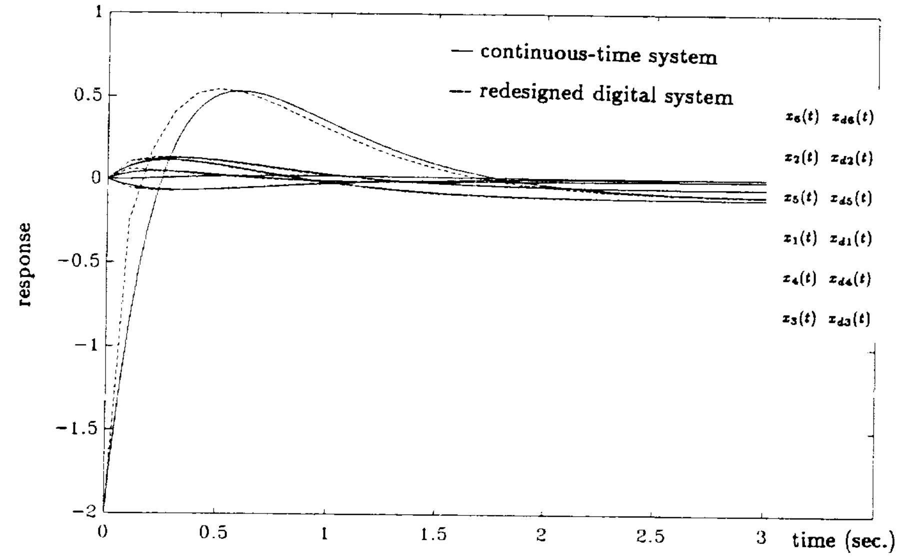

FIG. 4a. Simulation results.

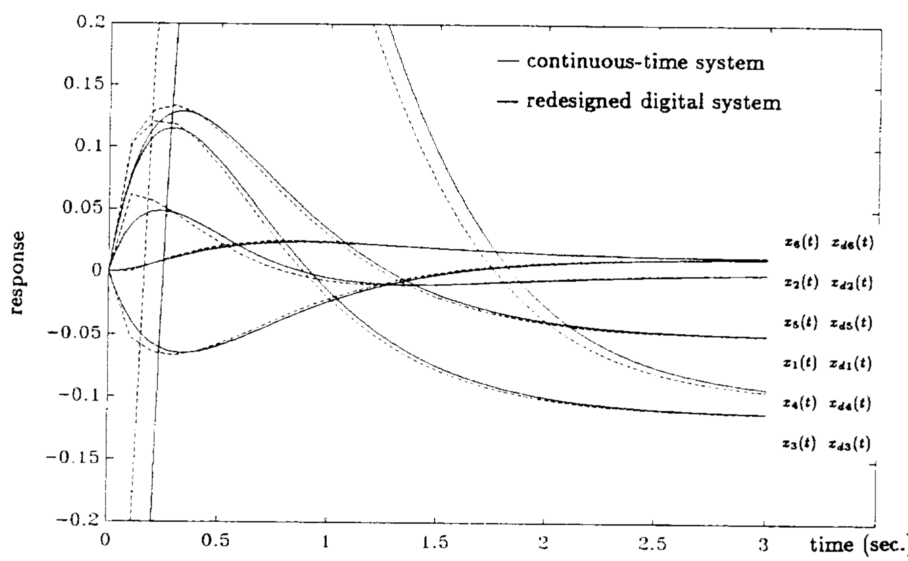

FIG. 4b. A part of Fig. 4a.

order regular subsystem and a nondynamic subsystem. Otherwise, we must apply a preliminary feedback control law, which is simpler and more efficient than that of Cobb's, to the singular system in order to eliminate all impulsive modes. Finally, we apply the given results to the reduced-order regular subsystem, and then by

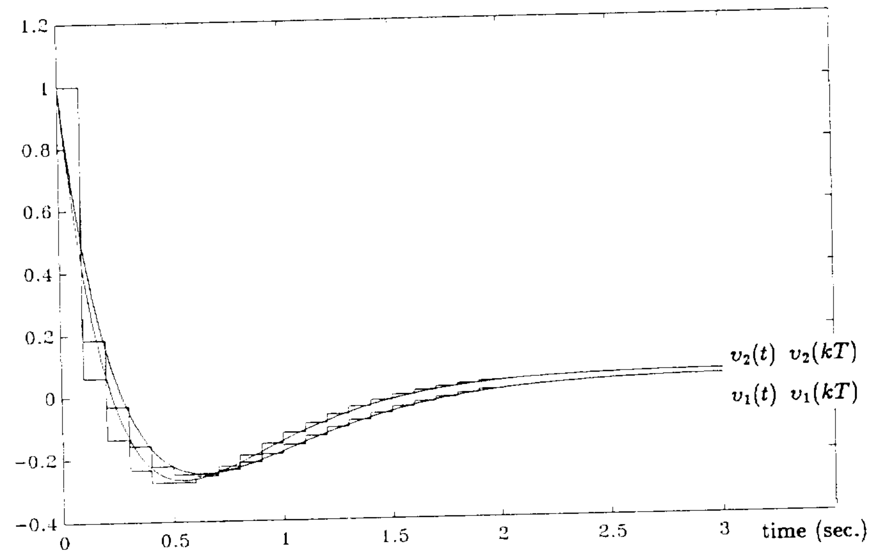

FIG. 5. Control signals.

relating the reduced-order regular subsystem are transformed back to those of the original coordinates.

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